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From Intuition to Tolerance: The Development of Carnap's Philosophy of Mathematics*

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1. INTRODUCTION

I shall consider a sequence of episodes in Carnap's thinking about mathematics beginning with his doctoral dissertation *Der Raum* (1922), which represents the most explicitly Kantian episode in his thought. I proceed to developments leading up through the *Aufbau* (1928), then turn to *The Logical Syntax of Language* (1934)—where the Principle of Tolerance is first officially formulated—and conclude by considering developments in the following semantic period and beyond.

I am not describing a transition from an epistemological grounding of mathematics in some kind of intuition—whether Kantian, Husserlian, Brouwerian, or Gödelian—to a pluralistic approach involving a "tolerant" appreciation of a variety of epistemological strategies. Nothing could be further from the truth. For Carnap is just as tolerant, in this sense, in *Der Raum*, the main point of which is to argue that logical and formalistic approaches to space and geometry (as in Russell or Hilbert), empirical or physical approaches (as in Helmholtz or Einstein), and intuition-based approaches in the Kantian, neo-Kantian, and Husserlian traditions are all basically correct. It is just that these different and apparently opposing approaches are really talking about different types or "meanings" of space—formal, physical, and intuitive. And, although Carnap abandons the intuitive space of *Der Raum* by the time of the *Aufbau*, he preserves important aspects of Kantian and neo-Kantian epistemology in this work and even later.

We obtain a better sense of Carnap's characteristic approach to the philosophy of mathematics—and to philosophy more generally—if we take seriously his claim that, throughout his career, he held no philosophical positions in the usual sense at all. Carnap develops this claim at length in his Intellectual Autobiography (1963a, 17-18). He describes the variety of "philosophical languages" he adopted in speaking with different "philosophically interested" friends (17): "With one friend I might talk in a language that could be characterized as realistic or even materialistic; . . In a talk with another friend, I

* I am especially indebted to very helpful comments on earlier versions by William Demopoulos and Wilfried Sieg.

might adapt myself to his idealistic kind of language. . . With some I talked a language which might be labeled nominalistic, with others again Frege's language of abstract entities of various types, . . . , a language which some contemporary authors call Platonic." He describes the emergence of what he came to call his philosophical neutrality about such matters:

> When asked which philosophical position I myself held, I was unable to answer. I could only say that my general way of thinking was closer to that of physicists and of those philosophers who are in contact with scientific work. Only gradually, in the course of the years, did I recognize clearly that my way of thinking was neutral with respect to the traditional controversies, e.g., realism vs. idealism, nominalism vs. Platonism (realism of universals), materialism vs. spiritualism, and so on. (17-18)

Carnap concludes by relating this general philosophical attitude to the official Principle of Tolerance that he first formulates explicitly in *Logical Syntax* (18): "This neutral attitude toward the various philosophical forms of language, based on the principle that everyone is free to use the language most suited to his purpose, has remained the same throughout my life. It was formulated as 'principle of tolerance' in *Logical Syntax* and I still hold it today, e.g., with respect to the contemporary controversy about a nominalist or Platonic language."

As I shall argue in detail, the suggestion that the Principle of Tolerance merely makes explicit a conception that Carnap held throughout his life is misleading, and it instead signals a radically new direction in his thought. Nevertheless, if we glance briefly at some of the debates in the philosophy of mathematics with which he engaged, we see that Carnap's claim to be committed to no philosophical positions in the usual sense is correct. For example, during the early 1930s, when he was working on *Logical Syntax* and involved in discussions with Gödel, Carnap's approach—especially in comparison with Gödel's—can be described as nominalist rather than Platonic. For Carnap there presupposes that the only scientifically meaningful "existence assertions" are those concerning physical objects in space and time. The entire set-theoretic (or type-theoretic) hierarchy of natural numbers, sets of natural numbers, sets of sets of natural numbers, and so on is merely a convenient device for setting up a mathematically tractable coordinate system for representing whatever spatio-temporal physical magnitudes (mass, charge, the electro-magnetic field, and so on) there happen to be: there is no substantive question concerning whether one or another mathematical representative of such magnitudes "really exists."[1] Yet when Carnap is responding, in "Empiricism, Semantics, and Ontology" (1950), to the nominalist philosophical predilections of Goodman, Quine, and Tarski, he rather defends the system of "abstract entities" (natural numbers) assumed in standard Peano arithmetic. That is, Carnap defends what he characterizes as a "Platonic" as opposed to "nominalistic" form of language in the passage (1963a, 17-18) quoted above.[2]

2. FROM *DER RAUM* TO THE *AUFBAU*

I begin with *Der Raum*—which, as suggested, involves Carnap's most explicit appeal to the Kantian conception of spatial intuition. It represents the final product, in particular, of Carnap's studies at the University of Jena with the neo-Kantian philosopher Bruno Bauch:

> I studied Kant's philosophy with Bruno Bauch in Jena. In his seminar, the *Critique of Pure Reason* was discussed in detail for an entire year. I was strongly impressed by Kant's conception that the geometrical structure of space is determined by the form of our intuition. The after-effects of this influence were still noticeable in the chapter on the space of intuition in my dissertation, *Der Raum*. (1963a, 4)

The first chapter, on formal space, begins with an exposition of formal logic following Carnap's most important teacher at Jena, Gottlob Frege, and then turns, as suggested, to an exposition of both axiomatic and what we now call set-theoretic (or type-theoretic) developments of the purely formal structure of space (where, for Carnap, the Russellian theory of relations is especially prominent). But it is in the second chapter, on intuitive space, that Carnap attempts to generalize the original Kantian (globally) Euclidean structure of our form of outer (spatial) intuition in order to accommodate the use of non-Euclidean geometry of variable curvature in Einstein's general theory of relativity.

The centerpiece of this second chapter is Carnap's modification of Hilbert's axiomatization of Euclidean geometry in *Grundlagen der Geometrie* (1899). Carnap

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[1] See Carnap (1934/1937, §38a), "On Existence Assumptions in Logic," which depends on the idea that Carnap's mathematical languages (Languages I and II) are "coordinate-languages" rather than "name-languages"—where the former are particularly important for formulating the "physical language" (§82). Compare also the discussion of the formal mode of speech in §79, where contentious philosophical assertions about various mathematical entities (natural numbers and real numbers) are translated into purely syntactic assertions about expressions in different formal languages.

[2] Carnap (1950, note 5) rejects the suggestions in Bernays (1935) and Quine (1948) that the use of classical mathematical methods is in any way associated with "Platonic metaphysics." For the background to Carnap (1950) in discussions involving Carnap, Goodman, Tarski, and Quine at Harvard in the academic year 1940-1941 see Friedman (2006).

takes Hilbert's axioms of incidence and order correctly to describe the necessary a priori structure of any "small" (local) region of space, as immediately presented to us in Husserlian *Wesenserschauung*.[3] When it comes to metrical structure, by contrast, Carnap reformulates Hilbert's axioms of congruence and parallels via a limiting procedure, so that specifically Euclidean geometry, following Riemann, is valid only in the very "smallest" regions—that is, infinitesimally.[4] The global structure of space, finally, can be determined only by what Carnap calls "postulates" (*Forderungen*), which stipulate how the local and infinitesimal structures fit together within a single "comprehensive" spatial structure. Carnap chooses these postulates in such a way as to allow maximal (but continuous) variability in the curvature from point to point, so as thereby to prepare the way for the accommodation, discussed at length in the third chapter, of Einstein's use of a (semi-)Riemannian geometry of variable curvature to describe physical space(-time).[5]

Before proceeding to this third chapter, however, I should observe that there is a significant mistake in Carnap's modification of Hilbert's axioms. For Hilbert's axioms of incidence and order are not even locally valid, in general, in a three-dimensional Riemannian manifold. These axioms characterize the *projective* structure of (three-dimensional) Euclidean space—which is precisely the structure given by the incidence and order relations of points and straight lines. And, in general, whereas a neighborhood of any point in such a Riemannian manifold is *topologically* equivalent to a neighborhood of (three-dimensional) Euclidean space, the two neighborhoods are not, in general, projectively equivalent. Whereas the (local) projective structure in question is common to all spaces of constant curvature (positive, negative, or zero), only the (local) topological structure is common to all (three-dimensional) manifolds of arbitrary curvature (constant or variable). There is thus no alternative, in the end, but to take Hilbert's axioms of incidence and order, just like his axioms of congruence and parallels, to be valid only infinitesimally.[6]

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[3] Carnap cites Husserl (1913) in connection with both *Wesenserschauung* and *Anschauung* more generally—and suggests (mistakenly in my view) that the original Kantian conception of *Anschauung* may include Husserlian *Wesenserschauung*.

[4] Carnap cites Riemann (1867), as well as the later (1919) edition due to Hermann Weyl.

[5] Carnap cites Einstein (1916), (1917), (1921).

[6] Hilbert's Axiom I, 6 states that if two points of a (straight) line lie in a plane, then every other point of the line lies in the same plane. Consider an arbitrary three-dimensional Riemannian manifold and a neighborhood of a given point *p*. To construct a "plane" through *p* in this neighborhood, we choose a two-dimensional subspace of the tangent space at *p* and then generate a two-dimensional hypersurface via a family of geodesics ("straight lines") through *p* tangent to the chosen two-dimensional subspace of the tangent space at *p*. Consider two different such geodesics through *p* and two points *q* and *r* located on these two different geodesics at a small distance from *p* (within the neighborhood in question). There will always exist (locally) a unique geodesic through *q* and *r*, but it will not lie, in general, on the original "plane" (hypersurface) through *p*. Carnap's mistake appears to derive from an ancestor of *Der Raum* that he submitted in connection with Bauch's seminar in March 1920, where, following Russell (1897), Carnap only considered spaces of constant curvature and thus envisioned a purely *projective* generalization of the

Returning now to the third chapter of *Der Raum*, Carnap conceives intuitive space as something like a Kantian form of intuition in the original sense, which in particular, is supposed to frame all perceptual experience of physical objects. Carnap here introduces a central distinction between freely chosen or "optional" (*wahlfrei*) form and "necessary" (*notwendig*) form, where only (local) topological structure comprehends the facts of perceptual experience within necessary form and thereby presents them *uniquely*—independently of the freely chosen metrical (and "straightness") stipulations that are then laid down by convention. In this way, the immediate deliverances of Husserlian *Wesenserschauung* provide our perceptual experience with the structure of a generalized Kantian form of intuition, where the intuitive space in question is compatible with all possible Riemannian metrical structures.[7] In order to find a single spatial structure embracing all of these possible ("optional") determinations, therefore, we adopt (n-dimensional) topological space but no particular one of the infinity of possible (n-dimensional) metrical spaces.[8] Whereas necessary form comprehends the topological spatial features that are intrinsically present in our form of (spatial) intuition, optional form comprehends the additional (metrical and projective) spatial characteristics that are not built into our a priori (spatial) intuition—but are nonetheless indispensable for fully describing the facts of experience mathematically.[9]

Kantian conception of a form of intuition. See Carus (2007, chapter 4) for discussion of this earlier dissertation in relation to *Der Raum*.

[7] By emphasizing the overriding importance of *topological* structure, Carnap has now formulated an adequate representation of the relationship between his generalization of the Kantian conception of a form of intuition and the physical space(-time) of Einsteinian general relativity: compare note 6 above, together with the paragraph to which it is appended. Moreover, Carnap is clear on the three-fold relationship of subordination between topological, projective, and metrical structure throughout *Der Raum*. That he nonetheless falls into error in his modification of Hilbert's axioms in the second chapter is perhaps understandable in light of the fact that only nine months had elapsed between the completion of his earlier dissertation in March 1920 and the submission of *Der Raum* in January 1921—during which period Carnap had engaged, among other things, in extensive study of both Husserlian phenomenology and general relativity.

[8] For Carnap, we are only presented in Husserlian *Wesenserschauung* with spatial structures that are *at least* three-dimensional—the exact number of dimensions must then be determined by a combination of experience and convention.

[9] See the final paragraph of *Der Raum* (1922, 67): "It has been frequently discussed, by mathematicians as well as philosophers, that Kant's contention concerning the significance of space for experience is not shaken by the theory of non-Euclidean spaces, but must be transferred from the three-dimensional Euclidean system, the only one known to him, to a more general one. But to the question which this is to be, the answers are either indeterminate, as only isolated properties of the three-dimensional Euclidean structure are proposed as requiring generalization, or contradictory, chiefly because of a failure to distinguish the different meanings of space and insufficient clarity about the conceptual interrelations among the kinds of space themselves—especially the relation of the metrical to the superordinate topological ones. According to the above reflections, the Kantian conception must be endorsed. The spatial system possessing experience-constituting significance, in place of that suggested by Kant, can be precisely specified as topological intuitive space with indefinitely many dimensions. With that, not only the attributes of this system, but at the same time those of its order framework [i.e., the corresponding formal space—MF], are declared to be conditions of the possibility of any object of experience whatsoever."

This generalized Kantian distinction between necessary and optional form also figures centrally in "Dreidimensionalität des Raumes und Kausalität" (1924). Carnap begins by distinguishing between "experience of the first level" and "experience of the second level"—between "primary" and "secondary" worlds—in so far as the first is subject to a univocal or "necessary formation" (*notwendige Formung*), whereas the second subjects this univocal structure to a further (non-univocal) "re-formation" (*Umformung*). The distinction depends, in particular, on "the necessity (*Notwendigkeit*) of the forms of the first level and the freely chosen character (*Wahlfreiheit*) of the forms of the second, which is manifested by the presence of different types of secondary worlds" (1924, 109)—the most important of which are the various mathematical forms for the world of physics considered in "Über die Aufgabe der Physik" (1923). Yet the 1924 distinction between primary and secondary worlds introduces an important new element not present in *Der Raum*: a transition from two to three dimensions in the respective spaces of the two worlds. The space exhibiting necessary form is not three-dimensional intuitive-cum-physical space (as in *Der Raum*), but rather the two-dimensional topological space of the visual field. Carnap (1924) thereby introduces a new epistemological problem—depicting the route from private subjective experience to the external physical world—which will be taken up and developed in the *Aufbau*.[10]

In the *Aufbau*, however, there is no remaining trace of the notion of *necessary* (perceptual) form and thus no remaining trace of either Kantian spatial intuition or the synthetic a priori (1928, §179): "After the first task, that of the constitution of objects, follows as *second the task of investigating the remaining*, non-constitutional *properties* and relations of the objects. The first task is solved by a stipulation, the second, by contrast, through *experience*. (According to the conception of constitutional theory there are no other components of cognition besides these two: the conventional and the empirical; and thus no synthetic a priori.)" What Carnap has in mind is that every constituted object is supposed to be set-theoretically (or type-theoretically) *defined* in a

[10] The central argument of "Dreidimensionalität" is that the secondary world is introduced in order to secure a "univocal determination" (*eindeutige Bestimmung*) of processes by universal causal laws, which is otherwise absent from the primary world—here lies the connection between the two "fictions" in Carnap's subtitle. This idea recurs in the *Aufbau* where Carnap is explaining the relative advantages of the different possible constitutional systems. Whereas the "idealistic" system with autopsychological basis correctly reflects the order of "epistemic primacy" (1928, §54), the "materialistic" system with physical basis reflects the standpoint of "empirical science" (§59): "A *materialistic constitutional system* has the advantage that it possesses the only domain as basis-domain (namely the physical) whose processes have a univocal lawfulness [*eindeutige Gesetzmäßigkeit*]." Carnap alludes to this tolerant attitude towards the choice of different possible bases in the passage from his Autobiography (1963, 17), cited in section 1 above, where he explains that it was only while working on the *Aufbau* that he "became aware that in talks with various friends I had used different philosophical languages, adapting myself to their ways of thinking and speaking."

finite number of steps on the basis of the elementary experiences comprising the first level of the iterative hierarchy of constituted objects. Once the object has thus been defined there remains only the infinite but *empirical* task of scientifically investigating its (non-constitutional) properties:

> According to the conception of the *Marburg School* (compare Natorp [1910] 18ff.) the object is the eternal X, its determination is an incompletable task. In opposition to this it is to be noted that finitely many determinations suffice for the constitution of the object—and thus for its univocal definite description [*eindeutige Kennzeichnung*] among the objects in general. Once such a definite description is set up the object is no longer an X, but something univocally determined—whose complete description [*Beschreibung*] then certainly still remains an incompletable task. (ibid.)

Thus Carnap's rejection of the synthetic a priori involves a modification of the "genetic" (*erzeugend*) conception of cognition characteristic of the Marburg School of neo-Kantianism—whose most important representatives, for Carnap, are Paul Natorp and Ernst Cassirer.[11]

I say "modification" because Carnap does not reject this version of neo-Kantianism completely. On the contrary, and in accordance with his general attitude of tolerance concerning epistemological matters in the *Aufbau* (compare note 10 above), he explicitly acknowledges what he takes to be the significant epistemological theses that he shares with this School (1928, §177): "Constitutional theory and *transcendental idealism* agree in representing the following position: all objects of cognition are constituted (in idealistic language, are 'generated in thought [*im Denken erzeugt*]'); and, moreover, the constituted objects are only objects of cognition *qua* logical forms constructed in a determinate way." That Carnap has the Marburg School specifically in mind here is confirmed by (1928, §5). And, although he rejects the Marburg commitment to the synthetic a priori, it is clear that Carnap agrees with the "genetic" conception—against what he takes to be overly crude versions of empiricism—that the characterization of empirical objects via *logical* forms is essential to their proper (scientific) cognition.[12]

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[11] The reference to Natorp (1910) is explicit in the above quotation, and it becomes clear elsewhere that Cassirer (1910) is equally important to Carnap: see note 12 below.

[12] Compare (1928, §75): "The merit of having discovered the necessary basis of the constitutional system thereby belongs to two entirely different, and often mutually hostile, philosophical tendencies. *Positivism* has stressed that the sole *material* for cognition lies in the undigested experiential *given*; here is to be sought the *basic elements* of the constitutional system. *Transcendental idealism*, however, especially the neo-Kantian tendency (Rickert, Cassirer, Bauch), has rightly emphasized that these elements do not suffice; *order-posits* must be added, our 'basic relations'." Carnap is here referring to Cassirer (1910) and Bauch (1923). Bauch's neo-Kantianism was that of the Southwest School, in which he was trained by Heinrich

In sum, the only a priori experience-constituting elements that Carnap now accepts are formal-logical. And for space, in particular, the only a priori such elements that he now accepts are those which, in *Der Raum*, he took to belong to *formal* space. Thus, in the *Aufbau*, Carnap constructs what he calls the perceptual world, and then the world of physics, by beginning with R^4 (1928, §125), embedding the sensory qualities in the autopsychological realm into this world—as it were along spatial lines of sight (§§126-134)—and finally assigning mathematical representatives of various physical magnitudes (scalar fields, vector fields, and so on) to points in R^4 via the "physico-qualitative coordination" (§136). Moreover (and in accordance, in this respect, with the conception of *Der Raum*), only the topological structure of R^4 counts as experience-constituting in Carnap's construction, since both the division of R^4 into (representatives of) three-dimensional space and one-dimensional time and the metrical properties of both space and time remain to be determined by a combination of experience and convention. Nevertheless, whereas space and time, for Carnap, are thereby represented by purely formal (purely analytical) mathematical entities (in terms of a spatio-temporal coordinate system), they are by no mean identified with such purely formal entities. On the contrary, space and time themselves are constitutionally introduced as essentially perceptual and in this sense intuitive—as what we might call *perceptual* space and time. *Physical* space and time are then parasitic on perceptual space and time. For Carnap, in the *Aufbau*, there therefore remains a crucial asymmetry between his conception of arithmetic and analysis, on the one side, and his conception of geometry (both spatial and spatio-temporal), on the other, and this asymmetry, as we shall see, persists throughout his career.[13]

3. SYNTAX AND THE PRINCIPLE OF TOLERANCE

Carnap's serious engagement with what we now call the philosophy of mathematics—and, in particular, with what we now call the foundations of mathematics—begins in the next phase of his career, culminating in the publication of

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Rickert at Freiburg. For further discussion of the Marburg and Southwest Schools in relation to Carnap see Friedman (1999, chapter 6), (2000).

[13] In §6 of his Autobiography on "The Foundations of Mathematics" Carnap begins by characterizing his general conception in terms of analyticity but explicitly excludes geometry (1963a, 49): "In the foregoing, the term 'mathematics' is meant to include the theory of numbers of various kinds and their functions, furthermore abstract fields, e.g., abstract algebra, abstract group theory and the like, but to exclude geometry." In thus taking only arithmetic, analysis, and algebra to be essentially formal or analytic disciplines, while geometry remains essentially intuitive or synthetic, Carnap follows Frege. In taking geometry to be synthetic *a posteriori*, however, Carnap instead follows Einstein: compare note 27 below, together with the paragraph to which it is appended.

Logical Syntax in 1934. This phase begins, as Carnap makes clear, with a famous lecture Brouwer presented at Vienna in 1928:

> In the Circle we also made a thorough study of intuitionism. Brouwer came to Vienna and gave a lecture on his conception, and we had private talks with him. . . The empiricist view of the Circle was of course incompatible with Brouwer's view, influenced by Kant, that pure intuition is the basis of all mathematics. On this view there was, strangely enough, agreement between intuitionism and the otherwise strongly opposed camp of formalism, especially as represented by Hilbert and Bernays. But the constructivist and finitist tendencies of Brouwer's thinking appealed to us greatly. (1963a, 49)

One can well understand why Carnap and the Vienna Circle were attracted to these "constructivist and finitist tendencies," since they dovetailed nicely with the conception of empirical verification that the Circle found in both Wittgenstein's *Tractatus* and in the *Aufbau*.[14] Despite these attractive features of Brouwer's intuitionism, however, neither the Circle nor, especially, Carnap could remain content with it.

The most important problem, especially in Carnap's case, was that to go along with Brouwer was to become a partisan in the "foundations crisis" that embroiled some of the leading logicians and mathematicians of the time. Here we were faced with concerns about the consistency of classical mathematics in the wake of the logical and set-theoretical paradoxes that were addressed by three mutually opposing schools of thought. Formalism, represented by Hilbert and Paul Bernays, attempted to justify classical mathematics by developing a formal consistency proof using only limited finitist methods. Intuitionism, by contrast, represented by Brouwer and Arend Heyting, mounted a revolutionary challenge to classical mathematics based on a rejection of the law of excluded middle applied to quantifications over infinite totalities. And logicism, finally, attempted to preserve as much as possible of the original Frege-Russell conception of mathematics in the context of the more radical proposals of intuitionism and formalism. Carnap himself represented the logicist position at a defining symposium that he organized (in cooperation with Hans Reichenbach) on the foundations of mathematics at Königsberg in September 1930—where the other two viewpoints were represented by

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[14] See Carnap (1928, §180): "We presuppose in constitutional theory that it should be in principle possible to cognize whether or not a given basic relation holds between two given elementary experiences. But any state of affairs is composed out of nothing but . . singular relation-extension statements; and here the number of elements at issue in the basic relations, namely, the elementary experiences, is finite. From this it follows that it is in principle possible to establish in finitely many steps whether or not the state of affairs in question obtains." Compare the preceding §179 on verification.

Heying and John von Neumann.[15] Even here, however, Carnap clearly indicates, at the end of his contribution, that logicism, in his sense, needs to incorporate the valid insights of both formalism and intuitionism.

As is well known, the following discussion of the foundations of mathematics ended with Gödel's initial public announcement of his (first) incompleteness theorem.[16] Carnap reports that, in the month before this symposium, he had a stimulating discussion with Gödel about his epoch-making result:

> I often talked with Gödel about these problems [in metamathematics]. In August 1930 he explained to me his new method of correlating numbers with signs and expressions. Thus a theory of the forms of expressions could be formulated with the help of the concepts of arithmetic. He told me that, with the help of this method of arithmetization, he had proved that any formal system of arithmetic is incomplete and incompletable. When he published this result in 1931, it marked a turning point in the development of the foundations of mathematics.
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> After thinking about these problems for several years, the whole theory of language structure and its possible applications in philosophy came to me like a vision during a sleepless night in January 1931, when I was ill. (1963, 53)[17]

The project of *Logical Syntax* was now underway, and it is this project, of course, that initiates Carnap's own distinctive contribution to the debate on the foundations of mathematics.

In conformity with the metamathematical methods of Hilbertian proof theory, Carnap views any formulation of logic and mathematics as a syntactically described formal system, where the notions of well-formed formula, axiom, derivation, theorem, and so on can all be syntactically expressed.[18] In light of Gödel's recently published

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[15] The contents of this symposium appeared in the second volume of *Erkenntnis*: see Carnap (1931), Heyting (1931), von Neumann (1931). Translations of these papers lead off the well-known selection of readings by Paul Benacerraf and Hilary Putnam in both editions (1964), (1983); this volume, in turn, decisively influenced the development of philosophy of mathematics in the second half of the twentieth century.

[16] See Hahn, *et al*. (1931), where Gödel's contribution to the discussion occupies pp. 147-8; at the very end of the discussion, in response to a remark by von Neumann, Gödel presents the upshot of his first incompleteness theorem. The printed version concludes with a "Postscript" added by Gödel at the invitation of the editors (i.e., Reichenbach and Carnap), where he sketches the results of Gödel (1931)—which now include both the first and second incompleteness theorems. For a translation of (and introduction to) Gödel's contribution to the symposium (including the Postscript) see Feferman, *et al*. (1986, 196-205).

[17] This account by Carnap is historically misleading, since, as Wilfried Sieg has emphasized to me, Gödel did not yet have an arithmetization of syntax in elementary number theory at the time of the Königsberg meeting. Gödel was rather working within a system of type theory or set theory, where, in particular, finite sequences of numbers appear as higher-order objects; he only discovered the number-theoretic encoding of such sequences used in Gödel (1931) shortly after the Königsberg meeting: see Sieg (2013, 163-4)

[18] Carnap cites Hilbert (1926), (1931), Bernays (1930).

incompleteness results, however, Carnap does not pursue the Hilbertian project of constructing a proof of the consistency of classical mathematics using finitist means acceptable to the intuitionist.[19] Instead, he formulates both a system conforming to the strictures of intuitionism (Language I, a version of primitive recursive arithmetic) and a much stronger system adequate for full classical mathematics (Language II, a version of higher-order type theory over the natural numbers as individuals). For both systems, moreover, he defines a notion of logical truth (analyticity) intended syntactically to express the essential independence of such truth from all factual content. Finally, and most importantly, Carnap officially formulates, for the first time, the Principle of Tolerance:

> *In logic there are no morals*. Everyone may construct his logic, i.e., his form of language, as he wishes. He must only, if he wishes to discuss it with us, clearly indicate how he wishes to do it, and give syntactical rules instead of philosophical considerations. (1934, §17)

Thus, in particular, both types of system (intuitionist and classical) should be syntactically described and investigated, and the choice between them, if there is one, should then be made on practical or pragmatic grounds rather than prior purely philosophical commitments.

Directly following his discussion of Brouwer's visit to Vienna in his Autobiography, Carnap presents a succinct description of the *Syntax* view:

> According to my principle of tolerance, I emphasized that, whereas it is important to make distinctions between constructivist and non-constructivist definitions and proofs, it seems advisable not to prohibit certain forms of procedure but to investigate all practically useful forms. It is true that certain procedures, e.g., those admitted by constructivism or intuitionism, are safer than others. Therefore it is advisable to apply these procedures as far as possible. However, there are other forms and methods which, though less safe because we do not have a proof of their consistency, appear to be practically indispensable for physics. In such a case there seems to be no good reason for prohibiting these procedures so long as no contradictions have been found. (1963a, 49)[20]

[19] Carnap cites Brouwer (1913), (1929), Heyting (1930), (1930a), (1931).

[20] Carnap is representing "constructivism" or "intuitionism" by his Language I (primitive recursive arithmetic) and not, e.g., by (first-order) Heyting arithmetic. Gödel (1933) shows that the latter theory has the same consistency strength as classical (first-order) Peano arithmetic—a result on which, to the best of my knowledge, Carnap never comments.

And, as we know, the Principle of Tolerance becomes central to Carnap's philosophy from this point on. It is important to appreciate, however, that, despite what Carnap suggests in the relevant passage from his Autobiography (1963a, 18) cited in section 1 above, this Principle goes far beyond the more general "neutral attitude" towards opposing philosophical positions that he had maintained throughout his life.

To begin with, modern logic and mathematics had furnished the uncontentious neutral framework, for Carnap, in all of his work up until now. In *Der Raum*, as we have seen, he appealed to the logic of Frege and Russell in his chapter on formal space, to Hilbert's axiomatization of geometry in his chapter on intuitive space, and to the Riemannian theory of mathematical manifolds of any number of dimensions both here and in his chapter on physical space. Similarly, the logic of *Principia Mathematica*, together with elements of what we now call Zermelo-Fraenkel set theory, constitutes the unquestioned formal framework for the *Aufbau*.[21] Indeed, Carnap's confidence in *Principia Mathematica*, at that time, was so strong that he confounds contemporary readers of the Preface to the first edition by appealing to the contradictions that had precipitated the foundations crisis in order to praise modern—Frege-Russell—logic at the expense of "the traditional logic" (*die alte Logik*).[22] In *Logical Syntax*, however, it is precisely the modern logic developed by Frege and Russell—together with the foundations that they had attempted to provide for modern (classical) mathematics—which is now at issue. So it is entirely unclear what possible neutral framework for pursuing scientific philosophy and avoiding hopeless philosophical disputes can still be available.

I shall return to the question of neutrality in section 4 below. But I meanwhile want to note a further radically new element in Carnap's *Syntax* approach. The Frege-Russell tradition had operated with a revision of the original Kantian distinction between analytic and synthetic propositions appropriate to the new logic. Whereas Kant had appealed to

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[21] In the *Aufbau* (and later) Carnap tends not to distinguish between type theory and set theory; in connection with the latter he was particularly close to Fraenkel. The discussion of the axiomatic method in Fraenkel (1923, §13) stimulated Carnap's interest in questions concerning the completeness of axiom systems, and the resulting exchanges with Fraenkel are reflected in the greatly expanded discussion in the fifth chapter of Fraenkel (1928), which contains a number of references to Carnap's work, including the *Aufbau*, which itself refers to Fraenkel (1923). The Axiom of Replacement is explicitly formulated only in Fraenkel (1928, §16.9), but is already informally discussed (with reference to his original paper of 1922) in Fraenkel (1923, §13); Carnap discusses Replacement—and cites Fraenkel (1928)—in *Syntax* (1934, §33). For the close relationship between Carnap and Fraenkel see Reck (2007, §§II-III), which also discusses Carnap's own work in "general axiomatics" during this period, including its influence on Gödel's eventual formulation of both his completeness and incompleteness results.

[22] See Carnap (1928/1967, xv): "In the last few decades mathematicians have developed a new logic. They were forced to do this by necessity, namely by the foundations crisis of mathematics, in which traditional logic had proved an utter failure. It not only proved incapable of dealing with these difficult problems but something far worse happened, the worst fate that can befall a scientific theory: it led to contradictions."

both subject-predicate containment and following from the law of identity (together with definitions) as criteria for analyticity (A6-7/B10-11), only a modification of the second criterion—following from the laws of the new logic (together with definitions)—was appropriate now. If we could show that all of classical mathematics is derivable from the laws of the new logic of Frege and Russell, we would therefore have accomplished something quite substantial. For Carnap, however, the question of reducing mathematics to logic in some antecedently understood sense is now abruptly dismissed:

> Whether, in the construction of a system of the kind described, one admits only logical signs in the narrower sense as primitive signs (as in Frege and Russell) or also mathematical signs (as in Hilbert), and whether one lays down only logical sentences in the narrower sense as L-primitive sentences or also mathematical sentences, is not a question of philosophical significance but only one of technical suitability. In the construction of Languages I and II we have followed Hilbert in this respect and chosen the second procedure. Incidentally, the question is not even precisely posed; we have, to be sure, given a formal distinction between logical and descriptive signs in general syntax; but a precise division of the logical signs in our sense into logical signs in the narrower sense and mathematical signs has so far not been provided by anyone. (1934, §84)

Since, in the preceding paragraph, Carnap has stated that the "*requirement of logicism*" amounts merely to the demand that the language of mathematics be included in the total language of (empirical) science, it is clear that he is no longer concerned with logicism in the traditional sense at all—even in the attenuated sense of Carnap (1931).[23]

Finally, as Carnap suggests in the last quotation, the key to his new conception of logico-mathematical truth is the distinction between logical and descriptive signs. The basic idea is that analytic (L-valid) propositions are those that are valid independently of any descriptive signs that may appear in them and are thereby independent, in particular, of all empirical facts about the world. Analytic propositions, in this sense, are entirely empty of content, and, as a result, Carnap holds that logico-mathematical propositions are fundamentally instruments for inferentially organizing the (contentful) propositions of empirical science. Carnap asserts in his Autobiography that this conception "derives essentially from Frege" (1963a, 12), and he later indicates that Wittgenstein's *Tractatus*

[23] Compare note 15 above, together with the paragraph to which it is appended. Carnap (1931) begins by characterizing logicism in the traditional way: the concepts of mathematics are to be defined in terms of logical concepts; the theorems are to be derived from logical axioms. He discusses each of these in turn in the following two sections (§§I-II) and finds technical difficulties with both, the most important of which (in both cases) concern "the problem of impredicative definitions" (§III)—for which Carnap then sketches an "attempt towards a solution" in the final section (§IV). See Goldfarb (1996) for discussion of both this solution and the radically different situation in *Logical Syntax*.

provided the crucial insight into emptiness of "factual content" (25). Once again, however, Carnap's conception radically diverges from the tradition of Frege and Russell, including the Wittgenstein of the *Tractatus*. For, not only is the formal characterization of the distinction between logical and descriptive signs to which Carnap refers in §84 language relative and therefore language variable, but it is also subject to technical difficulties and is quickly abandoned in his semantic period—where he is basically content to explain the distinction, in the case of any particular language, simply by enumeration of the logical signs.[24] So it appears entirely unclear, once again, what the force of Carnap's new conception of logico-mathematical truth is supposed to be.

4. CARNAP'S ANTI-FOUNDATIONAL NEUTRALITY

The key to understanding Carnap's new conception is to appreciate the extent to which it has no philosophical commitment to any foundational program—whether logicist, formalist, or intuitionist. Rather, according to precisely the Principle of Tolerance, the point of viewing the statements of logic and mathematics as analytic lies solely in our *freedom to choose* which system of logic and mathematics best serves the formal deductive needs of empirical science. Classical mathematics, for example, is much easier to apply, especially in physics, than intuitionist mathematics, while the latter, being logically weaker, is less likely to result in contradiction.[25] The choice between the two systems is therefore purely practical or pragmatic, and it should be sharply separated, in particular, from all philosophical disputes about what mathematical entities "really are" (independent Platonic objects or mental constructions, for example) or which such entities "really exist" (only natural numbers, for example, or also real numbers, that is, sets of natural numbers). Carnap aims to use the new tools of metamathematics definitively to dissolve all such philosophical disputes once and for all, and to replace them with the much more fruitful project of language planning—which, as Carnap

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[24] Carnap presents his characterization in (1934, §50); for its technical problems see Creath (1996), Awodey (2007); for the transition from syntax to semantics see Ricketts (1996), Awodey (2007).

[25] Compare the passage on the Principle of Tolerance (1963a, 49) quoted in the previous section, together with the appended note 20 above. In *Foundations of Logic and Mathematics* (1939)—the most important work on this topic in his semantic period—Carnap explains the point more fully (1939, §20, 192-3): "Concerning mathematics as a pure calculus there are no sharp controversies. These arise as soon as mathematics is dealt with as a system of 'knowledge'; in our terminology, as an interpreted system. Now, if we regard interpreted mathematics as an instrument of deduction within the field of empirical knowledge rather than as a system of information, then many of the controversial problems are recognized as being questions not of truth but of technical expedience. The question is: Which form of the mathematical system is technically most suitable for the purpose mentioned? Which one provides the greatest safety? If we compare, e.g., the systems of classical mathematics and of intuitionistic mathematics, we find that the first is much simpler and technically more efficient, while the second is more safe from surprising occurrences, e.g., contradictions."

understands it, has no commitment whatever to any foundational or other epistemological program.[26]

But what about the "neutral attitude" towards philosophical problems that Carnap closely associates with the Principle of Tolerance (1963a, 18)? Can he really dissolve the philosophical disputes in question from a standpoint that is neutral with respect to the very points at issue? In particular, can he really dissolve the dispute between classical mathematics and intuitionism in an entirely non-question-begging way? Although it is not explicitly focussed on this particular dispute, the exchange between Carnap and E. W. Beth in Schilpp (1963) is especially illuminating in this regard—all the more so because it does explicitly involve the relation between the Principle of Tolerance and mathematical intuition.

The main criticism Beth develops is that the *Syntax* project requires what he calls "a non-formal, intuitive, interpretation" (1963, 477)—so that it is less purely formal, and also less unrestrictedly tolerant, than Carnap appears to realize. The crux of Beth's argument is that syntax is itself a kind of arithmetic (as becomes especially clear under Gödel numbering). And, viewed as an arithmetic, a Carnapian syntax language or metalanguage may then have non-standard models—containing non-finite numbers (non-finite sequences of expressions) extending beyond the standard numbers 0, 1, 2, . . (so that, in the case of syntax, there may be numerals extending beyond the standard finite ones, for example, or derivations may have more than a finite number of steps). Someone who understood Carnap's syntax language in accordance with such a non-standard model would systematically misunderstand his main inductive definitions and results; and so, Beth argues, Carnap must implicitly be assuming that the syntax language is understood in accordance with the standard model. But, Beth claims, we are thereby faced with what he calls "a limitation regarding the Principle of Tolerance" (1963, 479). For, although someone who understands Carnap's syntax language in the standard way (such as, presumably, Carnap himself) can understand someone who uses a non-standard interpretation, the perverse practitioner of syntax in accordance with a non-standard interpretation (whom Beth dubs "Carnap*") would not be able to understand the standard one—for this practitioner lacks precisely the standard understanding of the concept of *finiteness*.

Carnap, in his reply to Beth, accepts Beth's technical point. And, accordingly, Carnap accepts Beth's claim that "[w]e find in *Logical Syntax* also concepts which, though defined in a purely formal way, are clearly inspired by a non-formal

[26] See the discussion of "Language Planning" in (1963a, §11).

interpretation" (1963b, 928). Carnap suggests that he understands this idea in terms of the notion of an *interpreted* formal language or calculus in the sense of the semantic approach that he developed shortly after *Logical Syntax*. Indeed, in *Foundations of Logic and Mathematics* (note 25 above), Carnap applies this notion to describe what he calls the "*customary interpretation"* of Peano arithmetic—where the calculus being interpreted is based on the term ‘*b*’, the functor ‘′’, and the predicate ‘*N*’ (1939, §17, 182): "The *customary interpretation* of the Peano system may first be formulated in this way: ‘*b*’ designates the cardinal number 0; if ‘. . .’ designates a cardinal number *n*, then ‘. . .′’ designates the next one, i.e., *n*+1; ‘*N*’ designates the class of finite cardinal numbers. Hence in this interpretation the system concerns the progression of finite cardinal numbers, ordered according to magnitude." Carnap is here appealing to a construction of the Frege-Russell cardinals within higher-order logic presented in a previous section (§14), and, in any case, there is no doubt that Carnap is presupposing the standard understanding of the concept of *finiteness*, just as Beth suggests.

Before proceeding with the reply to Beth, it is important to appreciate the significance that Carnap attaches to the distinction between a pure (syntactic) calculus and its (semantic) interpretation. In his discussion of the interpretation of *geometrical* calculi, in particular, Carnap strongly maintains his asymmetrical attitude towards the cases of arithmetic and analysis, on the one side, and geometry, on the other:

> When we referred to mathematics in the previous sections, we did not mean to include geometry but only the mathematics of numbers and numerical functions. Geometry must be dealt with separately. To be sure, the geometrical calculi, aside from their interpretation, are not fundamentally different in their character from the other calculi and, moreover, are closely related to the mathematical calculi. That is the reason why they too have been developed by mathematicians. But the customary interpretations of geometrical calculi are descriptive, while those of the mathematical calculi are logical. (1939, §21, 193)

The "*customary interpretation*" of a geometrical calculus "consists of a translation into the physical calculus . . together with the customary interpretation of the physical calculus" (§21, 194). And physicists, in contrast to mathematicians, "are concerned with a theory of space, i.e., of the system of possible configurations and movements of bodies" (§22, 196). The great achievement of modern Einsteinian physics, Carnap continues, is decisively to have shown that geometry in its customary interpretation is thus synthetic but *not* a priori (§22, 197-8)—and so, as Einstein himself suggests, "[t]he Kantian doctrine is based on a failure to distinguish between mathematical and physical geometry" (§22, 198).[27]

Returning to the reply to Beth, Carnap applies his semantic conception of interpretation to the metalanguages used in both syntax and semantics (1963b, 929): "Since the metalanguage *ML* serves as a means of communication between the author and the reader or among participants in a discussion, I always presupposed, both in syntax and in semantics, that a fixed interpretation of *ML*, which is shared by all participants, is given. This interpretation is usually not formulated explicitly; but since *ML* uses English words, it is assumed that these words are understood in their ordinary senses." Of course, the imaginary case constructed by Beth violates precisely this presupposition (ibid.): "Carnap* does not use the metalanguage *ML*, but a language *ML** which, although it uses the same words and sentences, differs from *ML*, since some of the words and sentences have different meanings." Yet Carnap is completely untroubled by this because he is simply assuming that an unproblematic understanding of the standard model of arithmetic is encapsulated in ordinary mathematical usage. There is no deep mystery here—there is no need to puzzle over the question how we force an uninterpreted formal calculus to designate or refer to its intended model. We simply give the customary interpretation of this system in unproblematic antecedently understood terms of ordinary mathematical language; appealing to an "intuitive grasp" of the standard model adds nothing at all.

Carnap is also clear that there are similar cases of failure of communication more directly relevant to his use of the Principle of Tolerance in *Logical Syntax*:

> It seems to be obvious that, if two men wish to find out whether or not their views on certain objects agree, they must first of all use a common language to make sure that they are talking about the same objects. It may be the case that one of them can express in his own language certain convictions which he cannot translate into the common language; in this case he cannot communicate these convictions to the other man. For example, a classical mathematician

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[27] This discussion should be compared with Carnap's later lectures on *Philosophical Foundations of Physics*, which support his conception of the analytic/synthetic distinction with the example of the (general) theory of relativity (1966, 257): "[Einstein] saw clearly the sharp dividing line that must always be kept in mind between pure mathematics, with its many types of logically consistent geometries, and physics, in which only experiment and observation can determine which geometries can be applied most usefully to the physical world." It should also be compared with Carnap's remarks on the conception of analyticity that he says "derives essentially from Frege" in the passage cited in the paragraph to which note 24 above is appended. Carnap there concludes (1963a, 12) that "the nature of logic and mathematics can be clearly understood only if close attention is given to their application in non-logical fields, especially in empirical science." As we have seen, this emphasis on the application of mathematics in *empirical* science—especially physics—remains constant throughout Carnap's career, and it represents a centrally important aspect of the legacy he derives from Kant: for further discussion see Friedman (2006).

> is in this situation with respect to an intuitionist or, to a still higher degree, with respect to a nominalist. (1963b, 929-30)

Just as Carnap cannot communicate the standard interpretation of the concept of *finiteness* to Carnap*, the intuitionist cannot understand the classical interpretation of unbounded existential quantification over the natural numbers.

Does this, as Beth suggests, then imply a restriction or limitation of the Principle of Tolerance? It may at first appear that it does. For Carnap's application of the Principle to this case poses the question, in a (syntactic or semantic) metalanguage, whether to adopt the classical or intuitionist logical rules for a particular object language (e.g., the language of physics). We weigh the relative safety (from the possibility of contradiction) of the intuitionist rules against the greater fruitfulness and convenience (in physics) of the classical rules and then make our choice. But if the intuitionist cannot understand the rules of the classical framework—and cannot, in particular, understand the necessarily even stronger classical metalanguage in which we (syntactically or semantically) describe these rules—then it would appear that our entire procedure simply begs the question against the intuitionist. In no way do we have a neutral shared metaperspective for evaluating the two positions on an equal footing.

This argument is certainly tempting, and I myself have succumbed to the temptation more than once. I now think, however, that it misses the essence of Carnap's conception, which emerges particularly clearly in the reply to Beth.[28] Just as in the case of our understanding of the standard model of arithmetic, Carnap presupposes that classical mathematics, as it is ordinarily practiced, is well understood. Indeed, classical mathematics, for Carnap, is a model or paradigm of clear and exact—scientific—understanding, and there is no way, in particular, to raise doubts about our understanding of this framework on the basis of independent purely philosophical commitments. To be sure, the early twentieth-century "crisis" in the foundations of mathematics, especially in the context of Gödel's incompleteness results, has raised serious technical questions relevant to the consistency of the classical framework, and this is precisely why, for Carnap, we should now take intuitionism seriously. To take it seriously, however, means that we entertain the proposal, starting from within the classical framework, that we should weaken its rules to make inconsistency less likely. There is nothing in Carnap's position blocking a practitioner of classical mathematics from entertaining this option or

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[28] Two of my earlier discussions are reprinted in Friedman (1999, chapters 7 and 9). See Goldfarb and Ricketts (1992), Ricketts (2007) for alternative readings. My reading has now moved closer to theirs, but with an importantly different emphasis (more attuned to Carnap's own) on the role of logic and mathematics in *empirical* science.

even deciding then to adopt it. Carnap has therefore not begged the question about the choice between classical and intuitionist mathematics as he understands it. That a philosophically committed intuitionist cannot understand the choice as Carnap understands it is irrelevant, for the situation in which we find ourselves has arisen within the paradigmatically well-understood practice of classical mathematics itself.

In 1936, at the very beginning of his semantic period, Carnap published "Von der Erkenntnistheorie zur Wissenschaftslogik," the point of which was to argue that all epistemological projects, including his own in the *Aufbau*, must now be renounced as "unclear mixture[s] of psychological and logical components" (1936, 36).[29] So Carnap, at this point, is decisively and explicitly breaking with the entire epistemological tradition. *Wissenschaftslogik*, in particular, is in no way concerned with either explaining or justifying our scientific knowledge by exhibiting its ultimate basis (whatever this basis might be). It is rather concerned with developing a new role for philosophy vis-à-vis the empirical sciences that will maximally contribute to scientific progress by facilitating the development and clarification of new mathematical inferential frameworks for these sciences—while, at the same time, defusing all the traditional metaphysical disputes and obscurities that have constituted (and, according to Carnap, continue to constitute) serious obstacles to progress in both philosophy and the sciences. Carnap's philosophy of mathematics, in the end, is thus not only anti-foundational but also, in an important sense, anti-philosophical. For *Wissenschaftslogik*, in the famous words from *Logical Syntax* (1934, §72; bold emphasis added), "***takes the place** of the inextricable tangle of problems one calls philosophy*." Yet it is in precisely the way in which it emerges out of, and imaginatively responds to, a variety of foundational challenges faced by early twentieth-century thought—which, as we have seen, involve some of the deepest intellectual problems surrounding modern logic, mathematics, mathematical physics, and philosophy—that Carnap's project acquires its undoubted philosophical significance and force.

[29] This paper was presented at the International Congress for Scientific Philosophy in Paris in 1935—where, at Carnap's urging, Tarksi first presented his work on truth and semantics to a large philosophical audience. While other members of the Circle (especially Neurath) were skeptical, an enthusiastic embrace of Tarski's work is characteristic of Carnap's semantic period: see (1963a, §10). For further discussion of Carnap (1936) see Richardson (1996).

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